University of Crete
Department of Mathematics
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MEM-227 Field Theory
Problem Set 5

Problem 5.1 : Let K be the splitting field of $f(x) = x^4 - 25 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.
(i) Find the degree $[K : \mathbb{Q}]$.
(ii) Find the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
(iii) Find the subgroups $H \leq G$.
(iv) For every subgroup H, find its fixed field $\mathcal{F}(H)$.

Problem 5.2 : Let $\omega \in \mathbb{C}$ be a primitive 7-th root of unity.
(i) Find the degree $[\mathbb{Q}(\omega) : \mathbb{Q}]$.
(ii) Find the Galois group $G = \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$.
(iii) Find the subgroups $H \leq G$.
(iv) For every subgroup H, find its fixed field $\mathcal{F}(H)$.

Problem 5.3 : Let K be the splitting field of $f(x) = x^3 - 7 \in \mathbb{Q}[x]$ over $\mathbb{Q}$. Find K, compute the Galois group $\mathrm{Gal}(K/F)$ and show that it is isomporphic to S_3.

Problem 5.4 : Let K be a finite extension of F and for any $\alpha \in K$, definte the map

$$L_\alpha : K \to K, \quad L_\alpha(\gamma) = \alpha\gamma.$$

(i) Show that L_α is F-linear.
(ii) Show that $\min(F, \alpha) \mid \det(xI - L_\alpha)$.
(iii) Under what condition is $\min(F, \alpha) = \det(xI - L_\alpha)$?

Problem 5.5 : Let $\omega \in \mathbb{C}$ be a primitive 5-th root of unity.
(i) Show that $\deg\big(\min(\mathbb{Q}, \omega^2 + 1)\big) = 4$.
(ii) Compute $\min\big(\mathbb{Q}, \omega^2 + 1\big)$.

Problem 5.6 : Use problem 5.4 to compute $\min\Big(\mathbb{Q}, \sqrt{2} + \sqrt{3}\Big)$.