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MEM-227 Field Theory
Problem Set 4

Problem 4.1 : Which of the following extensions are normal?
(i) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$
(ii) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$
(iii) $\mathbb{Q}(\sqrt{2}+\sqrt{3})/\mathbb{Q}$
(iv) $\mathbb{Q}\left(\frac{\sqrt{2}+1}{\sqrt{3}}, \frac{\sqrt{3}-1}{2}\right)/\mathbb{Q}$
(v) $\mathbb{Q}(i\sqrt{2})/\mathbb{Q}$

Problem 4.2 : Let $\omega \in \mathbb{C}$ be a primitive 3rd root of unity. Which of the following extensions are normal?
(i) $\mathbb{Q}(\omega, \sqrt{2})/\mathbb{Q}$
(ii) $\mathbb{Q}(\omega\sqrt{2})/\mathbb{Q}$
(iii) $\mathbb{Q}(\omega, \sqrt{2})/\mathbb{Q}(\sqrt{2})$

Problem 4.3 : Let K_1/F and K_2/F be finite, normal extensions. Prove that K_1K_2/F and $K_1 \cap K_2/F$ are normal. Note: $K_1K_2 = K_1(K_2) = K_2(K_1)$ is the smallest extension of F that contains both K_1 and K_2.

Problem 4.4 : Let K/F be a finite extension and $F(a)/F$ be a normal extension, such that $K \cap F(a) = F$.
(i) Prove that $\min(F, a) = \min(K, a)$.
(ii) Prove that $[K(a) : F] = [K : F] \cdot [F(a) : F]$.

Problem 4.5 : Let $f \in F[x]$ be a polynomial of prime degree. Assume that for every extension K of F, if f has a root in K then f splits in K. Prove that either f is irreducible over F or it splits in F.

Problem 4.6 : Show that the hypotheses of the previous problem hold in the following polynomials $f \in F[x]$.
(i) $f(x) = x^p - a$, where $\text{char}(f) = p$,
(ii) $f(x) = x^p - x - a$, where $\text{char}(f) = p$,
(iii) $f(x) = x^p - a$, where $\text{char}(f) \neq p$ and F contains an element ω such that $\omega^p = 1$ and $\omega \neq 1$.

Problem 4.7 : Let F be a field of characteristic p and $a \in F \setminus F^p$. Prove that $x^p - a$ is irreducible over F.

Problem 4.8 : Let F be a field of characteristic p, K be an extension of F and $\alpha \in K$. Prove that a^{p^m} is separable over F for some $m \geq 0$.