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MEM-227 Field Theory
Problem Set 3

Problem 3.1 : Compute the the splitting field of each of the following polynomials over $\mathbb{Q}$
(i) $x^3 - 5 \in \mathbb{Q}[x]$
(ii) $x^6 - 2 \in \mathbb{Q}[x]$
(iii) $x^4 - 1 \in \mathbb{Q}[x]$
(iv) $x^3 - x^2 - x - 2 \in \mathbb{Q}[x]$

Problem 3.2 : Let F be a field, $f \in F[x]$ be an irreducible polynomial of degree 2 and let a be a root of f. Show that $F(a)$ is a splitting field of f over F.

Problem 3.3 : Find all $\mathbb{Q}$-embeddings $\mathbb{Q}\left(\sqrt[4]{2}\right) \to \overline{\mathbb{Q}}$

Problem 3.4 : Find all $\mathbb{Q}$-embeddings $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) \to \overline{\mathbb{Q}}$

Problem 3.5 : Let $\omega \in \mathbb{C}$ be a primitive 3-rd root of unity. Prove that $\mathbb{Q}\left(\sqrt[3]{2}\right)$ and $\mathbb{Q}\left(\omega\sqrt[3]{2}\right)$ are isomorphic.

Problem 3.6 : The complex 8-th roots of unity are the roots of $x^8 - 1 \in \mathbb{Q}[x]$. Let ζ be a primitive 8-th root of unity.
(i) Show that $\zeta^j, 0 \leq j \leq 7$, are the (distinct) roots of $x^8 - 1$.
(ii) Which of those roots are *primitive*?
(iii) Find the splitting field K of $x^8 - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.
(iv) Find all $\mathbb{Q}$-automorphisms $K \to K$.
Hint: First extend $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}$ to all possible $\sigma : \mathbb{Q}\left(\sqrt[8]{2}\right) \to \mathbb{Q}\left(\sqrt[8]{2}\right)$.
Then extend each σ to all possible $\tau : \mathbb{Q}\left(\sqrt[8]{2}, \zeta\right) \to \mathbb{Q}\left(\sqrt[8]{2}, \zeta\right)$.
Note that $x^4 + 1 = \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)$.

Problem 3.7 : Let p be a prime.
(i) Find a splitting field K of $x^p - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.
(ii) Show that $[K : \mathbb{Q}] = p(p-1)$.
(iii) Find all $\mathbb{Q}$-automorphisms $K \to K$.

Problem 3.8 : Let K, L be extensions of $\mathbb{Q}$ and $\sigma : K \to L$ an isomorphism. Prove that σ fixes $\mathbb{Q}$.

Problem 3.9 : Prove that the fields $\mathbb{Q}\left(\sqrt{2}\right)$ and $\mathbb{Q}\left(\sqrt{3}\right)$ are not isomorphic.