University of Crete
Department of Mathematics
and Applied Mathematics

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T. Garefalakis

MEM-227 Field Theory
Problem Set 2

Problem 2.1 :
(i) Compute $\left[\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) : \mathbb{Q}\right]$.
(ii) Compute $\min\left(\mathbb{Q}\left(\sqrt{2}\right), \sqrt{3}\right)$.
(iii) Show that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) = \mathbb{Q}\left(\sqrt{2} + \sqrt{3}\right)$ (Hint: $\left(\sqrt{2} + \sqrt{3}\right)\left(\sqrt{2} - \sqrt{3}\right) = -1$).
(iv) Compute $\min\left(\mathbb{Q}, \sqrt{2} + \sqrt{3}\right)$.

Problem 2.2 : Let a be a root of $f = x^3 + x + 1 \in \mathbb{Q}[x]$.
(i) Prove that f is irreducible over $\mathbb{Q}$.
(ii) Compute the degree of the extension $\mathbb{Q}(a)/\mathbb{Q}$ and find a $\mathbb{Q}$-basis of $\mathbb{Q}(a)$.
(iii) Write a^{-1} and $(a+1)^{-1}$ as a $\mathbb{Q}$-linear combination of the basis you found.

Problem 2.3 : Let p be a prime and $\omega = e^{2\pi i/p} \in \mathbb{C}$ be a primitive root of unity.
(i) Show that $[\mathbb{Q}(\omega) : \mathbb{Q}] = p - 1$.
(ii) Let $a = \mathrm{Re}(\omega) = \cos\left(\frac{2\pi}{p}\right)$. Show that $a \in \mathbb{Q}(\omega)$.
(iii) Show that $[\mathbb{Q}(\omega) : \mathbb{Q}(a)] = 2$.
(iv) Show that $[\mathbb{Q}(a) : \mathbb{Q}] = \frac{p-1}{2}$.

Problem 2.4 : Let $F \leq K$ be an algebraic extension and $a, b \in K$. Let $[F(a) : F] = n$, $[F(b) : F] = m$ and assume that $(n, m) = 1$.
(i) Show that $[F(a, b) : F] = n \cdot m$.
(ii) Show that $F(a) \cap F(b) = F$ (Hint: $F(a) \cap F(b)$ is a field. For any $c \in F(a) \cap F(b)$, what can you say about $[F(c) : F]$?)

Problem 2.5 : Let $F \leq R \leq K$, where K/F is an algebraic field extension, and R is a ring. Prove that R is a field. (Hint: Let $a \in R$, $a \neq 0$ and show that $F(a) \leq R$).