UNIVERSITY OF CRETE
DEPARTMENT OF MATHEMATICS
AND APPLIED MATHEMATICS

SPRING 2026
T. GAREFALAKIS

MEM-227 FIELD THEORY
PROBLEM SET 1

Problem 1.1 : Let $f = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ and $\frac{r}{s} \in \mathbb{Q}$ be a zero of f, with $r, s \in \mathbb{Z}$, $(r, s) = 1$. Prove that $r \mid a_0$ and $s \mid a_n$.

Problem 1.2 : Let p be a prime. Show that the polynomial $\frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$ is irreducible.

Problem 1.3 : Check the following polynomials for irreducibility in the respective polynomial ring:
(i) $x^2 + 1 \in \mathbb{F}_2[x]$,
(ii) $x^2 + x + 1 \in \mathbb{F}_2[x]$,
(iii) $x^2 + 1 \in \mathbb{F}_3[x]$,
(iv) $x^3 + x + 1 \in \mathbb{F}_5[x]$.

Problem 1.4 : Find all irreducible polynomials in $\mathbb{F}_2[x]$ of degree ≤ 4.

Problem 1.5 : Let F be a field. Adapt Euclids proof of the infinity of prime numbers, to prove that there are infinitely many irreducible polynomials in $F[x]$.

Problem 1.6 : For each of the following complex numbers, show that it is algebraic over $\mathbb{Q}$ and compute its minimal polynomial over $\mathbb{Q}$.
(i) $\sqrt{5}$,
(ii) $1 + \sqrt{2}$,
(iii) $\sqrt{1 + \sqrt{5}}$,
(iv) $\sqrt{2} + \sqrt{3}$.

Problem 1.7 : Let K/F be a field extension. Prove that $K = F \Leftrightarrow [K : F] = 1$

Problem 1.8 : Let K/F be an extension of prime degree p. If $F \leq L \leq K$ show that either $L = F$ or $L = K$.

Problem 1.9 : Let K/F be an extension of odd degree. Show that for any $a \in K$, $F(a) = F(a^2)$.
Hint: Note that $F(a^2) \leq F(a)$. What can $\min(F(a^2), a)$ be?

Problem 1.10 : Prove that $\mathbb{Q}\left(\sqrt{5} + \sqrt{7}\right) = \mathbb{Q}\left(\sqrt{5}, \sqrt{7}\right)$.

Hint: Compute $\left(\sqrt{5} + \sqrt{7}\right)^3$ and use it to show that $\sqrt{5}, \sqrt{7} \in \mathbb{Q}\left(\sqrt{5} + \sqrt{7}\right)$.