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CHARACTERIZING THE VAN HIELE LEVELS OF DEVELOPMENT IN GEOMETRY

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This study provides a description of the van Hiele levels of reasoning in geometry according to responses to clinical interview tasks concerning triangles and quadrilaterals. The subjects were 13 students from Grades 1 through 12 plus a university mathematics major. The tasks included drawing shapes, identifying and defining shapes, sorting shapes, determining a mystery shape, establishing properties of parallelograms, and comparing components of a mathematical system. The students' behavior on the tasks was consistent with the van Hieles' original general description of the levels, although the discreteness of levels, particularly of analysis and abstraction, was not confirmed. The use of formal deduction among students who were taking or had taken secondary school geometry was nearly absent, consistent with earlier observations by Usiskin (1982).

Many American mathematics educators first learned of the van Hiele model of development in geometry through the efforts of Wirszup (1976) in the early 1970s. For this study, the five van Hiele levels were initially thought of in the following way, using descriptions by Dina van Hiele (van Hiele–Geldof, 1957) as modified by Hoffer (1981):

Level 0 (Visualization). The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 1 (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.

Level 2 (Abstraction). The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level 3 (Deduction). The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.

Level 4 (Rigor). The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

This study was undertaken to investigate the following research questions:

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1. Are the van Hiele levels useful in describing students' thinking processes on geometry tasks?

2. Can the levels be characterized operationally by student behavior?

3. Can an interview procedure be developed to reveal predominant levels of reasoning on specific geometry tasks?

Other studies have sought information concerning the hierarchical nature of the levels and the assignment of students to levels (Mayberry, 1983). Some have measured the geometric abilities of students as a function of van Hiele level (Usiskin, 1982), and some have investigated the effects of instruction on a student's predominant van Hiele level (Fuys, Geddes, & Tischler, 1985).

METHOD

Developmental and Experimental Phases

The study reported here is the final experimental phase of a 3-year project to investigate the van Hiele levels in school geometry. The first year of the study was a developmental phase in which the experimental tasks, an interview script, and a protocol analysis coding packet were written and revised three times. Each revision took place after successive pilot interviews were conducted by the four researchers on the project. The researchers then conferred to suggest and compose revisions of the tasks, script, and packet that would facilitate both the administration of the interviews and the coding scheme for analyzing them. The goal was to obtain an interview script and accompanying analysis packet that could easily be administered by teachers and researchers. The interview script and analysis packet are included in the final report of the project (Burger, 1986) and can be obtained from the authors on request.

Sample

The subjects for the final experimental interviews consisted of 45 students from 5 school districts in 3 states: Corvallis, Oregon; Central Linn, Oregon; Eugene, Oregon; East Lansing, Michigan; and New Albany, Ohio. We originally intended to interview students in Grades 7–12, choosing students before, during, and after they had taken a high school geometry course. The responses of students during the interviews in the developmental phase, however, encouraged us to administer the tasks to much younger students and to college-age mathematics students in order to attempt to characterize the extremes of the van Hiele levels. Thus, the 45 subjects were drawn from 7 grade categories (0 to 6): early primary (Grades K–1), primary (Grades 2–3), middle (Grades 4–8), Algebra 1 (pregeometry), geometry, Algebra 2 (postgeometry), and college mathematics majors. A summary of the sample according to the grade categories and interview locations is given in Table 1.

With the exception of the college students, who were asked by one of the

Table 1
Frequency of Students in the Interview Sample by Grade Category and Location

	Location					
Grade category	Corvallis	Central Linn	Eugene	East Lansing	New Albany	Total
0. Grades K–1				2		2
1. Grades 2–3				4		4
2. Grades 4–8				6	4	10
3. Algebra 1 (Gr. 8–11; pregeom.)	5	3	4			12
4. Geometry (Gr. 9–12)	7	1			1	9
5. Algebra 2 (Gr. 10–12; postgeom.)	5				1	6
6. College Jr. (mathematics major)	2					2
Total	19	4	4	12	6	45

researchers to participate, the subjects were selected by their teachers. The researchers described the study and its purpose to the cooperating teachers and then asked the teachers to select some of their average-to-better students to participate in the interviews. The researchers specified the number of students of each sex in order to obtain about the same number of boys and girls in each grade category.

Interview Procedure

The experimental tasks were administered to each student by one of the four researchers in an audiotaped clinical interview. The students were told they were going to be asked some questions about geometric shapes. Pencils, paper, straightedge, and compasses were made available. The students were encouraged to use any of these implements at any time during the interview. The interviewer presented the tasks to each student in the same order according to the script. After each task had been completed, the interviewer was free to probe further or follow up on any response. The data for the study consisted of the audiotapes, the students' drawings, and the interviewers' notes.

The interviews were conducted in a separate room during the time of the student's mathematics class. Only the student and the interviewer were present. Each interview took from 40 to 90 minutes. Some of the younger children were not administered the tasks that involved formal geometric reasoning. Because some of the older students became quite involved in several of the tasks, several of the interviews had to be conducted in two sittings, 2 or 3 days apart.

Tasks

The interviews consisted of eight tasks dealing with geometric shapes. The tasks involved drawing shapes, identifying and defining shapes, sorting shapes, and engaging in both informal and formal reasoning about geometric shapes. The tasks were designed to reflect the descriptions of the van Hiele levels that were available in the literature (van Hiele, 1973; Wirszup, 1976) and to incorporate some of the ideas from the tasks that Dina van Hiele had administered to her own students in her research (Fuys, Geddes, & Tischler, 1984; van Hiele-Geldof, 1957). The drawing, identifying, and sorting tasks (six in all) were expected to tease out characterizations of van Hiele Levels 0–2 from the protocols. The first formal reasoning task was an inference game in which a particular type of shape was revealed by its properties. The second formal reasoning, task consisted of a series of questions about theorems, axioms, and proof. The formal reasoning tasks were intended to obtain data about Levels 2 and 3. No attempt was made to investigate van Hiele Level 4 with these subjects, a level that requires the ability to compare different geometries. Two sets of drawing, identifying, and sorting tasks were administered, one set for triangular shapes and one set for quadrilateral shapes. Examples of the tasks for one class of shapes are described below. The tasks for the other class of shapes were similar.

Drawing. The student was asked to draw a triangle, to draw another that was different from the first one in some way, to draw another that was different from the first two in some way, and so forth as long as the question proved fruitful. Then the student was asked how the figures differed and how many different triangles he or she could draw. This task investigated the properties that students varied to make "different" figures and explored whether they thought the number of possible triangles was finite or infinite.

Identifying and defining. Given a sheet of quadrilaterals (see Figure 1), the student was asked to put an *S* on each square, an *R* on each rectangle, and if he or she was familiar with the terms, a *P* on each parallelogram and a *B* on each rhombus. The student was asked to justify his or her markings and, if necessary, why some of the figures had been omitted. In the defining part of this activity, the student was asked, "What would you tell someone to look for in order to pick out all the rectangles on a sheet of figures? (The equivalent question was asked for the other familiar shapes.) Could you make a shorter list? Is No. 2 a rectangle? Is No. 9 a parallelogram?" Thus, this activity explored the student's definitions and class inclusions.

Sorting. A set of cutout triangles was spread out on the table (see Figure 2). The student was asked, "Can you put some of these together that are alike in some way? How are they alike? Can you put some together that are alike in a different way? How are they alike?" This line of questioning was continued as long as the student could come up with new sorting properties.

Mystery shape. This task was an inference game entitled "What's My Shape?" that the interviewer played with the student. The interviewer said,

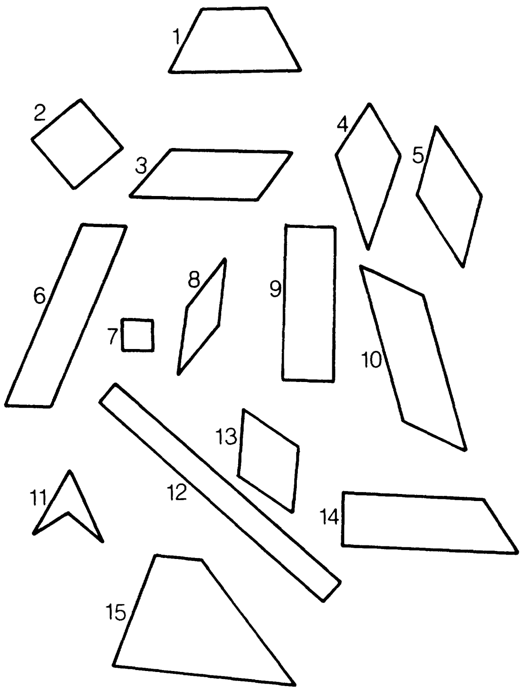

Figure 1. Quadrilaterals to be identified.

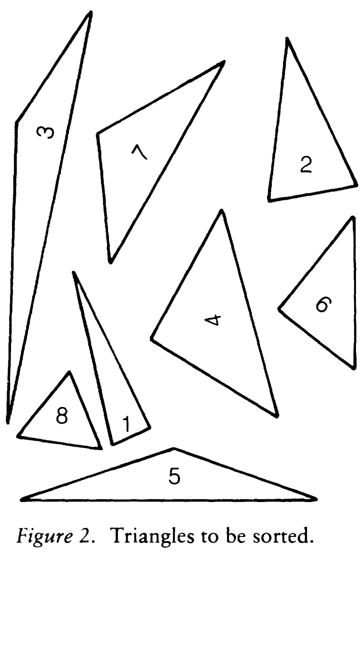

Figure 2. Triangles to be sorted.

"I'm going to show you a list of clues for a shape. I'll uncover the clues one at a time. When you have just enough clues to know for sure what the shape is, stop me. Otherwise ask for another clue. Feel free to use any of the drawing apparatus we have available." When the students indicated they had enough clues to decide the shape, they were asked how they knew with certainty and whether another clue would change their minds. This task elicited formal inference and addressed the role of necessary versus sufficient conditions to determine a shape. The list of clues for one of the shapes is given in Table 2.

Table 2
Clues for Parallelogram in "What's My Shape?"

1. It is a closed figure with 4 straight sides.
2. It has 2 long sides and 2 short sides.
3. The 2 long sides are the same length.
4. The 2 short sides are the same length.
5. One of the angles is larger than one of the other angles.
6. Two of the angles are the same size.
7. The other two angles are the same size.
8. The 2 long sides are parallel.
9. The 2 short sides are parallel.

Axioms, theorems, and a proof. This activity was conducted only with the secondary school and college students. They were asked if they had ever heard of the words *axiom, postulate,* or *theorem.* Then they were asked to give an example of each term with which they were familiar. Finally, they were asked to explore the following question and to explain their reasoning: "Suppose you had a quadrilateral with both pairs of opposite sides congruent. Must the opposite sides be parallel?" They were also asked the converse question.

Tape Pool

A "tape pool" of 14 of the 45 taped interviews was created for detailed investigation. The 14 tapes were chosen randomly within each grade category. Two tapes were randomly selected from each of Categories 1, 2, and 3; three tapes each from Categories 4 and 5; and one tape each from Categories 0 and 6. More tapes were selected from the higher grade categories because of the depth and variety of responses to the interview tasks by those students. A summary of the category, grade, sex, and interview site of the subjects in the tape pool is contained in Table 3.

Analysis and Coding Procedures

Three researchers reviewed each of the 14 tapes and completed a protocol analysis form for each one. The order of review of the 14 tapes was randomly selected and was different for each researcher. Response categories for each question in each task, which had been created during the developmental phase, were coded by the researchers. Excerpts from the tape transcripts that were particularly helpful in revealing the student's level of geometric thinking were transferred to the protocol analysis form. Each researcher was asked to

Table 3
Distribution of Students in the Tape Pool by Grade Category and Location

	Location					
Grade category	Corvallis	Central Linn	Eugene	East Lansing	New Albany	Total
0. Grades K–1				1F		1
1. Grades 2–3				3F 3M		2
2. Grades 4–8				4F 5M		2
3. Algebra 1 (pregeom.)	8F		9M			2
4. Geometry	10M 9F				12F	3
5. Algebra 2 (postgeom.)	11M 10F 10M					3
6. College Jr. (mathematics major)	15M					1
Total	7	0	1	5	1	14

Note. 1F stands for first-grade female.

assign a van Hiele level (0 to 3) on each task to each student. The level assigned was intended to represent the *predominant* level of thinking exhibited by the student on the task, that is, a preferred level of reasoning. Descriptions of the levels in the literature (van Hiele, 1973; Wirszup, 1976) helped to guide the coding.

After reviewing the entire tape, the researcher assigned an overall van Hiele level of reasoning to each student. Response summaries and anecdotal evidence to support the level assignment were recorded on the protocol analysis form. Also included in the analysis was a summary of any confounding evidence that seemed to conflict with the overall level assigned. Thus, for each of the 14 subjects, 3 different 8-dimensional vectors with van Hiele levels as entries were obtained, one vector for each researcher. The level assignment vectors were tested for interrater consensus with a scheme similar to that used by Mayberry (1983).

RESULTS

Illustrative Responses

From the pool of 14 students, a sample of 6 has been selected for reporting here. This sample was chosen for the variety of responses the students exhibited during the interviews, a variety that is representative of the pool. The students are Helen, a female 3rd grader from Category 1; Bud, a male 5th grader from Category 2; Amy, a female 8th grader from Category 3; Don, a male 10th grader from Category 4; Karen, a female 10th grader from Category 5; and Tom, a male university mathematics major, in his junior year, from Category 6. (These are not the students' real names.) The results are described for five tasks: drawing triangles, identifying quadrilaterals, sorting triangles, mystery shape, and axioms, theorems, and a proof.

On the drawing triangles activity, drawings by the young children often featured irrelevant attributes, such as the orientation on the page. Bud, for example, contrasted his triangles in the following ways (see Figure 3): Triangle 1 was "straight up"; Triangle 2 was "upside down"; Triangle 3 was "pointing that way [down]"; and Triangle 4 was "pointing that way [to the left]." Relevant attributes were often ignored, as in Triangle 5, which "has crooked lines." At first Bud thought there were about 12 different triangles, but later he said there were more than 1000.

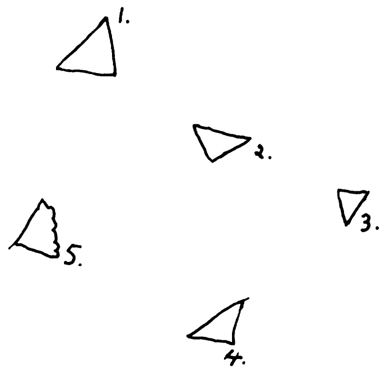

Figure 3. Bud's triangle drawings.

Amy drew her triangles carefully with a straightedge and contrasted them according to properties of their components. (See Figure 4.) She said that Triangle 2 "has a smaller angle than Number 1. Triangle Number 1 has a 45-degree angle. Triangle Number 2 has a 15-degree angle." Triangle 3 "has a wider angle than Number 1 and Number 2." Triangle 4 "has a 90-degree angle and a really small angle." Amy said there are three types of triangles but many different sizes and angles. The types are (a) three sides the same, (b) two sides the same, and (c) all sides different.

Some geometry students, such as Don, contrasted triangles according to general types. (See Figure 5.) In his drawings, Triangle 1 is equilateral, Triangle 2 is scalene, Triangle 3 is a right triangle, and Triangle 4 is isosceles. Don said there are at least five types of triangles: right, isosceles, equilateral, scalene, equiangular, and some combinations of these. For example, Triangle 3 is right scalene.

On the identifying and defining activity, the young children included many additional shapes among the squares and rectangles. For the shapes in Figure 1, Helen marked Shapes 2, 6, 7, 9, and 12 as squares. Bud marked 2, 4, 5, 7, 8, and 13 as squares. He marked 3, 6, 9, 10, and 12 as rectangles. Both students considered the shape's orientation on the page to be a relevant attribute. For

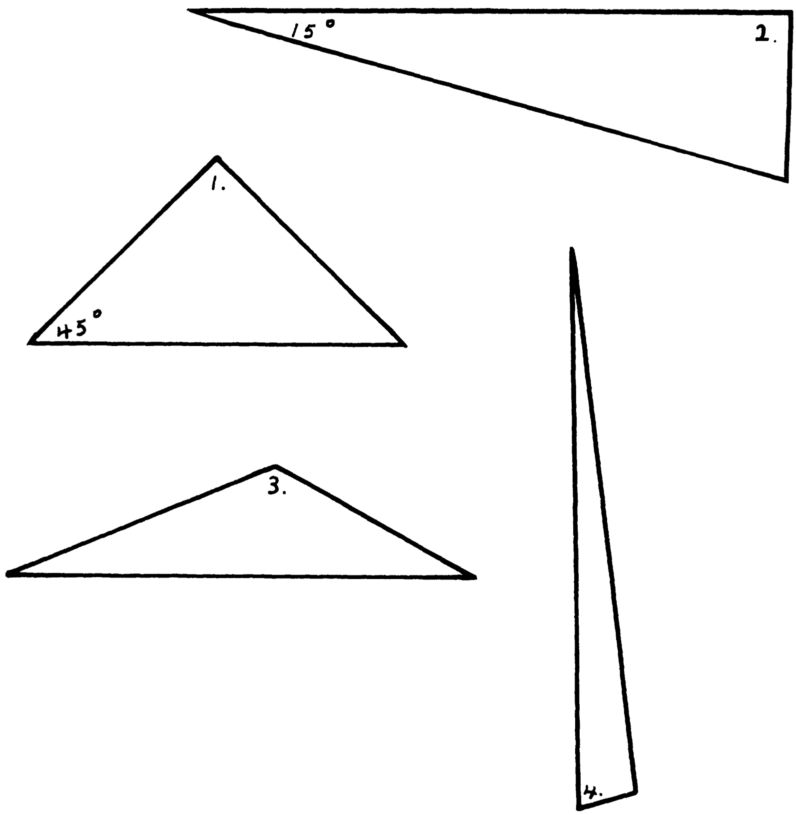

Figure 4. Amy's triangle drawings.

example, Shape 2, as shown, was not a square to Helen or Bud. Turned 45 degrees, it was. Some students, such as Amy, identified the types of quadrilaterals so as to prohibit class inclusions. To her, the squares in Figure 1 were Shapes 2 and 7; the rectangles were Shapes 9 and 12; the parallelograms were Shapes 3, 5, 6, and 10; and the rhombi were Shapes 8 and 13. In describing the shapes, she explicitly excluded squares from rectangles, saying that rectangles have "two sides equal and parallel to each other. Two longer sides are equal and parallel to each other, and they connect at 90 degrees." In defining parallelograms, she excluded rectangles and rhombi by saying that "two parallel lines the same length are connected by two slanting lines the same length. The slanting lines are [a] different length than the parallel lines."

Some postgeometry students, such as Karen, identified the shapes completely correctly and defined them by properties of their components, perhaps including some redundancies. For example, Karen defined a rectangle as a "four-sided closed figure, all angles are 90 degrees, and opposite sides are congruent." Don, however, defined the types of quadrilaterals by using

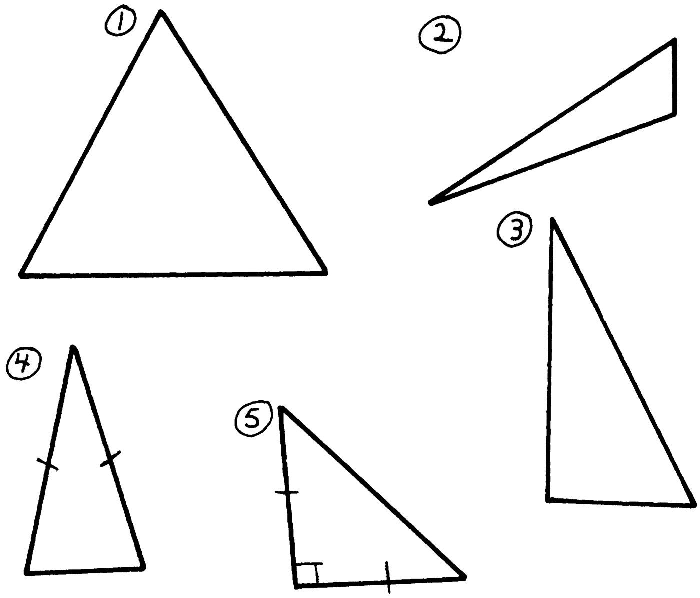

Figure 5. Don's triangle drawings.

relations among the types. To him, a square was "a parallelogram they has all the properties of a rhombus and a rectangle." A rectangle was "a parallelogram with at least one right angle." A parallelogram was "a quadrilateral with opposite sides congruent and parallel." Finally, to Don, a rhombus was "a parallelogram with two adjacent congruent sides." Tom, the university mathematics major, defined the various quadrilaterals independently of each other, then checked his definitions to be sure that they permitted the class inclusions he desired.

On the sorting activity, the young students made few sortings and often had difficulty differentiating them. For the shapes in Figure 2, Helen put Shapes 4, 5, 6, 7, and 8 together, saying they all were triangles. On her second sorting, she put Shapes 5 and 7 together, saying they both were triangles. Bud made two sortings, one consisting of Shapes 4 and 8, which he said had three equal sides. His other sorting consisted of Shapes 6 and 7, which he said had three unequal sides.

Amy sorted the triangles in many ways using properties of the sides and angles. She formed several partitions, using all the shapes, as opposed to collecting a subset with one common property. One of her partitions consisted of Shapes 1 and 6, which "have 90-degree angles," Shapes 2, 4, and 5, which "have two sides equal," and Shapes 3 and 7, which "have three different lengths and no 90-degree angles." (Shape 8 was inadvertently omitted during Amy's interview.) She also used some imprecise properties, such as "small angle at the top, wider angles at the base." Don sorted the triangles strictly by general types, using isosceles, scalene, and right triangles. Tom formed a variety of partitions based on precise properties of the sides and angles.

In determining the mystery shape (a parallelogram), the students made several types of arguments. The young children, such as Helen and Bud, treated the task strictly as a guessing game or used visual arguments, based on drawings, with no consistent use of the properties given by the clues. Other students, such as Amy and Karen, seemed to use the properties in an analytical way to confirm a guess, often using too few or too many clues. They thus seemed to be using the properties as necessary conditions. Don used the properties as part of a "casting out" strategy, eliminating certain general types of quadrilaterals as the clues dictated. Tom explicitly used a deductive strategy to determine the shape, building up his determination of the type formally from the clues.

On the last activity, axioms, theorems, and a proof, Karen contrasted the concept of postulate and theorem by saying that "a theorem is something you can prove. You can't prove postulates. A postulate is something that's assumed." When questioned, she stated that postulates are assumed, "because otherwise you couldn't have theorems. Theorems are important—but not for me." She admitted during a discussion with the interviewer that she was "not too logical" and preferred algebra ("because it's numbers") to geometry, especially to proofs, which she did not understand. Don remembered reading that there was a difference between theorems and postulates, but he could not recall the distinction. He said he never really understood the difference. Tom realized that theorems were proved from postulates and said that postulates are "so basic, they are accepted without proof." Many students, such as Amy, answered the question about parallelograms by making careful drawings and reasoning from their drawings. This strategy was common among the geometry and postgeometry students, such as Karen, who had no inclination to try to prove the result formally. Don and Tom chose to construct proofs of the proposition using properties of quadrilaterals, which they did successfully with some prompting. Tom had forgotten quite a bit of Euclidean geometry since high school and had to devise and prove some sufficient conditions for a quadrilateral to be a parallelogram. In fact, he formulated and logically tested many conjectures during his interview.

Assignment of Levels

Previous research studies have drawn some conclusions relevant to this study. Fuys et al. (1985) and Mayberry (1983) found that the van Hiele levels appear to be hierarchical in nature. Usiskin (1982) found that individual students can be assigned a van Hiele level but that students in transition from one level to the next are difficult to classify reliably. Mayberry (1983) also found that students can be on different levels for different concepts and that many students never reach the level of formal deduction, a conclusion shared by Usiskin (1982).

In this study, students were assigned to van Hiele levels of predominant reasoning by the three reviewers on each of the experimental tasks. The hierarchical nature of the levels, noted by Fuys et al. and Mayberry, was confirmed, as was the difficulty, observed by Usiskin, of assigning some students who appeared to be in transition between levels. That is, some disagreement among the reviewers occurred. Confounding data, cited by the reviewers as contributing to their difficulty in assigning a level, often served to bring them closer together. One reviewer might assign Level 1, for example, citing confounding data indicative of Level 0 while another reviewer would assign Level 0, citing confounding data indicative of Level 1.

In assigning students to levels, each reviewer had to provide data supporting the assignment. On the drawing tasks, students like Helen and Bud who varied visual qualities such as the orientation of the figure on the page or the "skinniness" of the figure and who thought there were only a finite number of triangles and quadrilaterals were assigned Level 0. Students like Amy who seemed to focus on the components of the shapes and who realized that the components could be varied in an infinite variety of ways were assigned Level 1. Students like Don who drew shapes as representatives of general types and who could interrelate shapes were generally assigned Level 2. The reviewers decided that the drawing tasks could not distinguish reasoning beyond Level 2.

On the identifying and defining tasks, the reviewers noted a number of imprecise visual qualities that some students used in describing the shapes. In addition, irrelevant attributes, such as orientation, were included in describing the shapes, and some relevant attributes were omitted. These responses were considered indications of Level 0 reasoning. References to visual prototypes ("a rectangle looks like a door") were common among students assigned Level 0 on the identification tasks. Students who contrasted shapes and identified them explicitly by means of their properties, as Amy did, were generally assigned Level 1 on the identification tasks. It was common for such students to prohibit class inclusions explicitly in describing the shapes and to recite a litany of their properties in defining them, far more than a minimal set of defining properties. Students such as Don who gave minimal characterizations of the shapes by using other types were assigned Level 2 on these tasks. Tom's frequent conjecturing and attempts to verify his conjectures by means of formal proof indicated a preference for Level 3 reasoning.

In analyzing the sorting tasks, the reviewers again detected a substantial amount of reasoning that used imprecise, visual qualities of the shapes. Sortings that were incomplete, as in Bud's case, were considered indicative of Level 0 reasoning because the student did not seem to be using the properties of the shapes *explicitly* in making the sortings. Sortings such as Amy's, done explicitly by using the properties of the shapes, were considered indicative of

Level 1 reasoning, even if the properties were imprecise. Sortings such as Don's and Tom's, which referred explicitly to a variety of types, were considered indicative of Level 2 reasoning. Again, the reviewers decided that the sorting tasks did not elicit reasoning beyond Level 2.

On the mystery shape task, the students who treated the task as a guessing game when asked to justify their decision on a shape and who made no use of the properties even as necessary conditions (e.g., ignoring the properties if they contradicted their guess) were assigned Level 0. Students such as Amy and Karen who seemed to use the clues as necessary properties to confirm a guess were generally assigned Level 1. Students such as Don who searched for a minimal set of clues and determined the shape by a "casting out" strategy, eliminating types of shapes as the clues indicated, were generally assigned Level 2. Students such as Tom who explicitly used deduction in determining the shape were assigned Level 3.

On the proof part of the last activity, students who attempted to answer the question by means of drawings alone and who could only rephrase the question as an assertion were generally assigned Level 0. Students who treated the problem as a "physics" problem and made a variety of drawings to test the validity of the proposition inductively were assigned Level 1. On this task, a number of students who had successfully completed a geometry course opted for this inductive procedure as their preferred method of solving the problem. Students, such as Don, who desired a deductive argument but were able to produce one only with persistent probing by the interviewer were assigned Level 2. Tom produced a deductive argument on his own, devising and proving sufficient conditions for a quadrilateral to be a parallelogram. He was the only student assigned Level 3.

It should be noted that assignments to levels did not seem to be strictly related to age or to grade category. For example, many students who had studied geometry formally were assigned Level 0 or 1 on tasks, not Level 2 or 3 as might have been expected. Karen was such a student. Detailed accounts of the results of all 14 tape-pool interviews can be found in the report by Burger (1986).

Level Indicators

The data supporting assignments to levels on the tasks can be summarized by the following level indicators:

Level 0

1. Use of imprecise properties (qualities) to compare drawings and to identify, characterize, and sort shapes.

2. References to visual prototypes to characterize shapes.

3. Inclusion of irrelevant attributes when identifying and describing shapes, such as orientation of the figure on the page.

4. Inability to conceive of an infinite variety of types of shapes.

5. Inconsistent sortings; that is, sortings by properties not shared by the sorted shapes.

6. Inability to use properties as necessary conditions to determine a shape; for example, guessing the shape in the mystery shape task after far too few clues, as if the clues triggered a visual image.

Level 1

1. Comparing shapes explicitly by means of properties of their components.

2. Prohibiting class inclusions among general types of shapes, such as quadrilaterals.

3. Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry, and so forth.

4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape.

5. Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles.

6. Explicit rejection of textbook definitions of shapes in favor of personal characterizations.

7. Treating geometry as physics when testing the validity of a proposition; for example, relying on a variety of drawings and making observations about them.

8. Explicit lack of understanding of mathematical proof.

Level 2

1. Formation of complete definitions of types of shapes.

2. Ability to modify definitions and immediately accept and use definitions of new concepts.

3. Explicit references to definitions.

4. Ability to accept equivalent forms of definitions.

5. Acceptance of logical partial ordering among types of shapes, including class inclusions.

6. Ability to sort shapes according to a variety of mathematically precise attributes.

7. Explicit use of "if, then" statements.

8. Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if p implies q and q implies r, then p implies r) and the law of detachment (modus ponens).

9. Confusion between the roles of axiom and theorem.

Level 3

1. Clarification of ambiguous questions and rephrasing of problem tasks into precise language.

2. Frequent conjecturing and attempts to verify conjectures deductively.

3. Reliance on proof as the *final* authority in deciding the truth of a mathematical proposition.

4. Understanding of the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, proof.

5. Implicit acceptance of the postulates of Euclidean geometry.

Interpretation of Levels

During the course of the study, several features of the levels emerged that we were not aware of initially. First, the levels appear to be complex structures involving the development of both concepts and reasoning processes applicable to many task environments. Kieren and Olson (1983) have used the level structure to analyze students' acquisition of concepts and reasoning abilities in the Logo environment, for example. Such a development seems highly dependent on instruction and much less dependent, if at all, on age. Second, although the van Hieles have theorized that the levels are discrete structures, this study did not detect that feature. The occasional difficulties that reviewers had in deciding between levels while making level assignments can be considered as evidence questioning the discrete nature of the van Hiele levels. Last, several postgeometry students seemed likely to have regressed a level on some of the activities since their study of geometry. Some students exhibited different preferred van Hiele levels of reasoning on different tasks. Some even oscillated from one level to another on the same task under probing by the interviewer. This oscillation was particularly evident among some of the Category 5 students, like Karen, who seemed to regress from Level 2 to Level 1 as their predominant level of reasoning on the tasks. Flashes of Level 2 reasoning would occur but usually only as a result of probing. Such students, left to their own devices, seemed to prefer the relative safety of Level 1 reasoning and tended to avoid deduction, even though they knew it was available.

Thus, the levels appear to be dynamic rather than static and of a more continuous nature than their discrete descriptions would lead one to believe. Students may move back and forth between levels quite a few times while they are in transition from one level to the next. Our data particularly support this phenomenon between Levels 1 and 2. We suspect that a similar phenomenon may exist when students are in transition from Levels 2 to 3, although we would need more data from college-age mathematics students to make such a case.

IMPLICATIONS

With reference to the three research questions investigated in this study, it does appear that the van Hiele levels are useful in describing students' thinking processes on polygon tasks. Reviewers were able to make level assignments on the tasks, much as Mayberry was, with little confounding data. Consequently, it would be appropriate to investigate students' responses on tasks involving other geometry concepts, such as measurement, transformations, congruence, and similarity. The level indicators derived from this study provide a first characterization of the van Hiele levels in terms of student behavior. There was agreement among the three reviewers that these indicators were accurate, but perhaps minimal, initial characterizations. Enhanced characterizations of the levels by specific student behavior, such as that given by the level indicators above, may be possible using other geometric tasks. In addition, the van Hiele model of development in geometry may well serve as a basis for constructivist teaching experiments in geometry, as described by Cobb and Steffe (1983).

We found that the development of an interview script and the accompanying protocol analysis form greatly facilitated the interview procedures and helped to structure the summary of large amounts of verbal data. We strongly recommend such a procedure to others considering any kind of clinical investigation involving audio or video recording. The pilot interviews and script development phases were essential to the development of a final interview script and analysis procedure.

In addition to further research considerations, there seem to be clear implications for teachers in the results of this study (Shaughnessy & Burger, 1985). For one, concept formation in geometry may well occur over long periods of time and require specific instruction. A number of the secondary school students interviewed had incomplete notions of basic geometric shapes and their properties. How these students might reason about shapes in a formal way was most unclear. This observation might explain some of the frustration students and teachers have with secondary school geometry courses: Students are not sufficiently grounded in basic geometric concepts and relations to "reinvent" Euclidean geometry. Memorization may be their only recourse. In this same vein, the notion of "meeting" or confronting a level, that van Hiele (1980) has described, may indeed be a real phenomenon in mathematics teaching. Students in the study who appeared to reason at different levels used different language and different problem-solving processes on the tasks. This phenomenon would also occur between a teacher and a student who are operating at different levels. As van Hiele has suggested, neither person could understand the other's reasoning, resulting in frustration and discouragement.

LIMITATIONS

In this clinical study, a relatively small sample of students representing a very broad range of ages was interviewed in depth on triangle and quadrilateral concepts. Four interviewers and three reviewers were used to conduct and analyze the interview data. Some disagreements occurred in the reviews, although many of these were mitigated when all the confounding data were revealed. The results of the study are descriptive, revealing aspects of the students' cognitive processes on the tasks. These results contribute to our general understanding of cognitive processes in geometry but say little about specific efforts to improve the processes themselves.

CONCLUSIONS

All three research questions can be answered in the affirmative. A response to the first can be based on the consensus among the reviewers in assigning students to levels on the interview tasks. Consistent behavior among students assigned to the same level on specific tasks can be summarized by the level indicators. These, in turn, help characterize the levels operationally. The success of the structured interview, using a specific script as a basis, enabled the reviewers to compare many students' responses to the same tasks. Tasks that involved a variety of environments in which the concepts were embodied (drawing, identifying from pictures, sorting, and solving abstract problems) revealed modes of reasoning about specific concepts that the reviewers could identify with confidence. Adapting the study's procedures to investigate other geometric concepts seems clearly appropriate.

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